

A LOOP GROUP METHOD FOR PROJECTIVE MINIMAL AND DEMOULIN SURFACES IN THE 3-DIMENSIONAL REAL PROJECTIVE SPACE

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ABSTRACT. For a surface in the 3-dimensional real projective space, we define two different Gauss maps, which are both quadrics in \mathbb{R}^4 , and call the first-order Gauss map and the second-order Gauss map, respectively. It will be shown that the surface is a projective minimal (resp. Demoulin) surface if and only if the second-order (resp. first-order) Gauss map is conformal and Lorentz harmonic. Moreover for a Demoulin surface, it will be shown that the first-order Gauss map can be obtained by the natural projection of the Lorentz primitive map into the 6-symmetric space. We also characterize projective minimal and Demoulin surfaces via a family of flat connections on the trivial bundle $\mathbb{D} \times \mathrm{SL}_4\mathbb{R}$ over a simply connected domain \mathbb{D} in the Euclidean 2-plane.

INTRODUCTION

Curves and surfaces in the 3-dimensional real projective space \mathbb{P}^3 were the central theme of differential geometry in 19th century. Especially, various transformations for a surface in \mathbb{P}^3 were introduced by Darboux, Demoulin, Titzeica, Godeaux, Rozet, Wilczynski, etc., and their properties were extensively studied. The most prominent features of the theory of transformations were the *Laplace sequence* and the *line/sphere congruences* of a surface. It is well known that Toda equations which discovered in the theory of integrable systems in 1970s had been already known as the periodic Laplace sequence, and the classical Darboux and Bäcklund transformations were defined by sphere congruences and tangential line congruences, respectively.

The Demoulin surface is characterized by the coincidence of general four Demoulin transformations of a surface, which are given by the envelopes of Lie quadrics. On the one hand, the projective minimal surface is defined by a critical point of the projective area functional. It is known that Demoulin surfaces give a special class of projective minimal surfaces. Moreover, using the Plücker embedding from \mathbb{P}^3 to \mathbb{P}^5 , Godeaux introduced an analogue of the Laplace sequence, the so-called *Godeaux sequence* of a surface in \mathbb{P}^5 . Then the surface is a Demoulin surface if and only if the Godeaux sequence is six periodic. For more details, we refer the readers to [11].

By using modern theory of integrable systems and differential geometry of harmonic maps, projective minimal surfaces and Demoulin surfaces were investigated in [7, 8, 2]. More

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precisely in [2], through the Plücker embedding from \mathbb{P}^3 to \mathbb{P}^5 projective minimal surfaces were characterized by Lorentz harmonicity of the conformal Gauss map, which takes values in a certain Grassmannian. In [8], the Demoulin surfaces were characterized by a certain Toda equation and the Bäcklund transformation of a Demoulin surface was constructed. Moreover, many classes of surfaces characterized by geometric properties were related to various integrable systems in [7].

In this paper, we study projective minimal surfaces and Demoulin surfaces via a loop group method. We first define two different Gauss maps for a surface, which are quadrics in \mathbb{R}^4 , and call the *first-order Gauss map* and the *the second-order Gauss map*, respectively. The first-order Gauss map and the second-order Gauss map have the first-order contact and the second-order contact to the surface, respectively. It will be shown that the second-order Gauss map is always conformal and the first-order Gauss map is conformal if and only if the surface is a Demoulin surface, see Proposition 1.2.

Then the Lorentz harmonicity of these two Gauss maps are studied. It will be shown that the second-order Gauss map is Lorentz harmonic if and only if the surface is a projective minimal surface, and the the first-order Gauss map is Lorentz harmonic if and only if the surface is a Demoulin or a projective minimal coincidence surface, see Theorems 2.1 and 2.3. We note that coincidence surfaces are simple examples of a class of surfaces which have nontrivial projective deformations, the so-called *projective applicable surfaces*. Since the target spaces of the Gauss maps are symmetric spaces, the Lorentz harmonic maps are also characterized by a family of flat connections on the trivial bundle $\mathbb{D} \times \mathrm{SL}_4\mathbb{R}$.

Combining the results in Proposition 1.2 and Theorem 2.3, we see that the first-order Gauss map is conformal Lorentz harmonic if and only if the surface is a Demoulin surface, see Corollary 2.5. Finally it will be shown that the first-order Gauss map of a Demoulin surface can be obtained by the natural projection of the Lorentz primitive map into the 6-symmetric space, see Corollary 2.6.

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1. PRELIMINARIES

1.1. Surfaces in \mathbb{P}^3 and the Wilczynski frames. The *canonical system* of a surface S in the 3-dimensional real projective space \mathbb{P}^3 is given as follows, [11, Section 2.2]:

$$(1.1) \quad f_{xx} = bf_y + pf, \quad f_{yy} = cf_y + qf,$$

where f is a lift of S in $\mathbb{R}^4 \setminus \{\mathbf{0}\}$, b, c, p and q are functions of x and y , and the subscripts x and y denote the partial derivative with respect to x and y , respectively. Let $f = (f^0, f^1, f^2, f^3)^t \in \mathbb{R}^4 \setminus \{\mathbf{0}\}$ and assume that $f^0 \neq 0$. Then the surface S is given by $S = \frac{1}{f^0}(f^1, f^2, f^3)^t$ and a straightforward computation shows that

$$S_{xx} = bS_y - 2(\log f^0)_x S_x \quad \text{and} \quad S_{yy} = cS_x - 2(\log f^0)_y S_y.$$

This implies that x and y are asymptotic coordinates on S . Thus the coordinates (x, y) induce the Lorentz structure on the surface S . It is known that $8bc \, dx dy$ is an absolute invariant

symmetric quadratic form, which is called the *projective metric* and $8bc$ is called the *Fubini-Pick invariant* of a surface S . It is also known that the conformal class of $b dx^3 + c dy^3$ is an absolute invariant cubic form. It is known that a surface whose Fubini-Pick invariant $8bc = 0$ is ruled, thus we assume that $bc \neq 0$. Then the *Wilczynski frame* is defined as follows:

$$F = (f, f_1, f_2, \eta),$$

where

$$\begin{aligned} f_1 &= f_x - \frac{c_x}{2c}f, \quad f_2 = f_y - \frac{b_x}{2b}f, \\ \eta &= f_{xy} - \frac{c_x}{2c}f_y - \frac{b_y}{2b}f_x + \left(\frac{b_y c_x}{4bc} - \frac{bc}{2} \right) f. \end{aligned}$$

Then a straightforward computation shows that the Wilczynski frame F satisfies the following equations:

$$(1.2) \quad F_x = FU \text{ and } F_y = FV,$$

where

$$(1.3) \quad U = \begin{pmatrix} \frac{c_x}{2c} & P & k & bQ \\ 1 & -\frac{c_x}{2c} & 0 & k \\ 0 & b & \frac{c_x}{2c} & P \\ 0 & 0 & 1 & -\frac{c_x}{2c} \end{pmatrix}, \quad V = \begin{pmatrix} \frac{b_y}{2b} & \ell & Q & cP \\ 0 & \frac{b_y}{2b} & c & Q \\ 1 & 0 & -\frac{b_y}{2b} & \ell \\ 0 & 1 & 0 & -\frac{b_y}{2b} \end{pmatrix}.$$

Here we introduced functions k, ℓ, P and Q of two variables x and y as follows:

$$(1.4) \quad k = \frac{bc - (\log b)_{xy}}{2}, \quad \ell = \frac{bc - (\log c)_{xy}}{2},$$

$$(1.5) \quad P = p + \frac{b_y}{2} - \frac{c_{xx}}{2c} + \frac{c_x^2}{4c^2}, \quad Q = q + \frac{c_x}{2} - \frac{b_{yy}}{2b} + \frac{b_y^2}{4b^2}.$$

The compatibility conditions of (1.2) are

$$(1.6) \quad Q_x = k_y + k \frac{b_y}{b}, \quad P_y = \ell_x + \ell \frac{c_x}{c},$$

$$(1.7) \quad bQ_y + 2b_y Q = cP_x + 2c_x P.$$

These equations are nothing but the projective Gauss-Codazzi equations of a surface S . Since the traces of U and V are zero, the Wilczynski frame F takes values in $\text{SL}_4\mathbb{R}$ up to initial condition. From now on, we assume that the Wilczynski frame F takes values in $\text{SL}_4\mathbb{R}$.

1.2. Projective minimal surfaces and Demoulin surfaces. It is known that the *projective minimal surface* is defined by a critical point of the projective area functional:

$$\int bc dx dy,$$

where the functions b and c are defined in (1.1). Then the projective minimality can be computed as in [12]:

$$(1.8) \quad bQ_y + 2b_y Q = 0 \text{ and } cP_x + 2c_x P = 0,$$

where the functions P and Q are defined in (1.5).

The *Demoulin surface* is defined by the coincidence of general four Demoulin transformations of a surface, which are given by the envelopes of Lie quadrics. It is known that Demoulin surfaces are characterized by the functions P and Q in (1.5), see [11, Definition 2.8]:

$$(1.9) \quad P = Q = 0.$$

Remark 1.1. From the equations in (1.8) and (1.9), it is easy to see that Demoulin surfaces are projective minimal surfaces.

1.3. Two quadrics as Gauss maps. Let us use the following notation:

$$\text{diag}(a, b, c, d) = \begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix}, \quad \text{offdiag}(a, b, c, d) = \begin{pmatrix} & & & a \\ & & b & \\ & c & & \\ d & & & \end{pmatrix}.$$

Let S be a surface in \mathbb{P}^3 and F the corresponding Wilczynski frame defined in (1.2). We first define a map g_1 by

$$g_1 = F J_1 F^t,$$

where $J_1 = \text{offdiag}(1, 1, 1, 1)$. It is easy to see that g_1 maps to the space of symmetric matrices with determinant one and signature $(2, 2)$, which we denote by Q_1 . The special linear group $\text{SL}_4 \mathbb{R}$ transitively acts on this space by $gPg^t \in Q_1$ with $g \in \text{SL}_4 \mathbb{R}$ and $P \in Q_1$. Then the point stabilizer at J_1 is given by $K_1 = \{X \in \text{SL}_4 \mathbb{R} \mid XJ_1X^t = J_1\}$, which is isomorphic to the special orthogonal group with signature $(2, 2)$, which is denoted by $\text{SO}_{2,2}$. Thus Q_1 is isomorphic to the symmetric space $\text{SL}_4 \mathbb{R}/\text{SO}_{2,2}$:

$$(1.10) \quad g_1 : M \rightarrow Q_1 \cong \text{SL}_4 \mathbb{R}/K_1 = \text{SL}_4 \mathbb{R}/\text{SO}_{2,2}.$$

This map g_1 is known as a quadric which has the first order contact to the surface. Note that g_1 does not have the second order contact, see [9, Section 22]. We call g_1 the *first-order Gauss map* for a surface S in \mathbb{P}^3 .

We next define a map g_2 by

$$g_2 = F J_2 F^t,$$

where $J_2 = \text{offdiag}(1, -1, -1, 1)$. Similar to g_1 , it is easy to see that g_2 maps to the space of symmetric matrices with determinant one and signature $(2, 2)$, which we denote by Q_2 . The special linear group $\text{SL}_4 \mathbb{R}$ transitively acts on this space by $gPg^t \in Q_2$ with $g \in \text{SL}_4 \mathbb{R}$ and $P \in Q_2$. Then the point stabilizer at J_2 is given by $K_2 = \{X \in \text{SL}_4 \mathbb{R} \mid XJ_2X^t = J_2\}$, which is also isomorphic to $\text{SO}_{2,2}$. Thus Q_2 is also isomorphic to the symmetric space $\text{SL}_4 \mathbb{R}/\text{SO}_{2,2}$:

$$(1.11) \quad g_2 : M \rightarrow Q_2 \cong \text{SL}_4 \mathbb{R}/K_2 = \text{SL}_4 \mathbb{R}/\text{SO}_{2,2}.$$

This map g_2 is known as a Lie quadric which has the second order contact to the surface, see [9, Section 18]. We call g_2 the *second-order Gauss map* for a surface S in \mathbb{P}^3 . In [10], the second-order Gauss map g_2 was called the projective Gauss map.

Proposition 1.2. *The second-order Gauss map g_2 is conformal, and the first-order Gauss map g_1 is conformal if and only if the surface S is a Demoulin surface.*

Proof. We first introduce the inner product on the tangent space of Q_j , ($j = 1, 2$) as follows:

$$\langle X, Y \rangle_p = \text{Tr}(p^{-1} X p^{-1} Y), \quad X, Y \in T_p Q_j,$$

where p is a symmetric matrix of determinant one with signature $(2, 2)$. This inner product is invariant under the action of $g \in \mathrm{SL}_4\mathbb{R}$, since

$$\langle gXg^t, gYg^t \rangle_{gpg^t} = \mathrm{Tr}((gpg^t)^{-1}gXg^t(gpg^t)^{-1}gYg^t) = \langle X, Y \rangle_p.$$

Then a direct computation shows that

$$g_{2x} = 2F \mathrm{diag}(bQ, 0, -b, 0)F^t \quad \text{and} \quad g_{2y} = 2F \mathrm{diag}(cP, -c, 0, 0)F^t.$$

Thus

$$\langle g_{2x}, g_{2x} \rangle = \langle g_{2y}, g_{2y} \rangle = 0 \quad \text{and} \quad \langle g_{2x}, g_{2y} \rangle = \langle g_{2y}, g_{2x} \rangle = 4bc \neq 0.$$

Since the coordinates (x, y) are null for the conformal structure induced by S , the second-order Gauss map g_2 is conformal.

An another direct computation shows that

$$g_{1x} = 2F \begin{pmatrix} bQ & k & P & 0 \\ k & 0 & 0 & 1 \\ P & 0 & b & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} F^t, \quad g_{1y} = 2F \begin{pmatrix} cP & Q & \ell & 0 \\ Q & c & 0 & 0 \\ \ell & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} F^t.$$

Thus

$$\langle g_{1x}, g_{1x} \rangle = 16P, \quad \langle g_{1y}, g_{1y} \rangle = 16Q \quad \text{and} \quad \langle g_{1x}, g_{1y} \rangle = \langle g_{1y}, g_{1x} \rangle = 8(k + \ell) + 4bc.$$

Since the coordinates (x, y) are null for the conformal structure induced by S , the first-order Gauss map g_1 is conformal if and only if $P = Q = 0$. \square

1.4. (Lorentz) Harmonic and (Lorentz) primitive maps into (k) -symmetric spaces. It is known that the loop group method can be applied to harmonic maps from surfaces into symmetric spaces, see [3, 5]. Let M and N be a 2-manifold and a semisimple symmetric space, respectively and φ a map from M into N . We denote the symmetric space N as quotient G/K with semisimple Lie group G and closed subgroup K of G such that $(G_\sigma)_o \subseteq K \subset G_\sigma$, where $(G_\sigma)_o$ is the identity component of the fixed point group G_σ of the involution σ of the symmetric space N . Let Φ be the frame of φ taking values in G and $\alpha = \Phi^{-1}d\Phi$ the Maurer-Cartan form. According to the eigenspace decomposition of \mathfrak{g} with respect to the derivative of σ , that is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, we define α^λ as follows:

$$\alpha^\lambda = \alpha_{\mathfrak{k}} + \lambda^{-1}\alpha'_{\mathfrak{p}} + \lambda\alpha''_{\mathfrak{p}},$$

where $\alpha_{\mathfrak{k}}$ and $\alpha_{\mathfrak{p}}$ denote the \mathfrak{k} - and \mathfrak{p} -parts, and α' and α'' denote the $(1, 0)$ - and $(0, 1)$ -parts, respectively.

Remark 1.3. For a Riemann surface M with conformal coordinates $z = x + iy$, the $(1, 0)$ - and $(0, 1)$ -parts denote dz and $d\bar{z}$ parts, respectively, and for a Lorentz surface M with null coordinates (x, y) , the $(1, 0)$ - and $(0, 1)$ -parts denote dx and dy parts, respectively.

The following theorem is a fundamental fact about (Lorentz) harmonic maps from surfaces into symmetric spaces, see [3, 5].

Theorem 1.4. *Let M be a surface and N a semisimple symmetric space. A map $\varphi : M \rightarrow N$ is a (Lorentz) harmonic map if and only if $d + \alpha^\lambda$ is a family of flat connections.*

If the target manifold N is a semisimple k -symmetric space ($k > 2$), then there does not exist a loop group formulation for general (Lorentz) harmonic maps from a surface into N as the above. Instead, we restrict our attention to a rather special kind of (Lorentz) harmonic maps, the *(Lorentz) primitive maps* in a k -symmetric space, so that the loop group formulation can be applied.

Definition 1. Let φ be a map from a surface M into a semisimple k -symmetric space $N = G/K$ with the order k automorphism σ ($k > 2$) and $\alpha = \Phi^{-1}d\Phi$ the Maurer-Cartan form of the frame Φ of φ . Moreover, let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{k-1}$ be the eigenspace decomposition of the complexification of \mathfrak{g} according to the derivative of σ and define $\mathfrak{g}_{i+kn} = \mathfrak{g}_i$ for $n \in \mathbb{Z}$. Then φ is called the *(Lorentz) primitive map* if the following conditions are satisfied:

$$(1.12) \quad \alpha' \text{ takes values in } \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \text{ and } \alpha'' \text{ takes values in } \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where \prime and $\prime\prime$ are the $(1, 0)$ - and $(0, 1)$ -parts, respectively.

The following is a basic fact about (Lorentz) primitive maps.

Proposition 1.5.

- (1) *A (Lorentz) primitive map into a semisimple k -symmetric space N ($k > 2$) is (Lorentz) equiharmonic, that is, it is (Lorentz) harmonic with respect to any invariant metric on N .*
- (2) *Let φ be a (Lorentz) primitive map into a semisimple k -symmetric space N , ($k > 2$), and $\pi : N \rightarrow G/H$ the homogeneous projection. Then $\pi \circ \varphi$ is (Lorentz) equiharmonic.*

Let φ be a primitive map into a semisimple k -symmetric space N ($k > 2$) and Φ the corresponding frame. Moreover, let α be the Maurer-Cartan form of Φ , $\alpha = \Phi^{-1}d\Phi$. Define α^{λ} as follows:

$$\alpha^{\lambda} = \alpha_0 + \lambda^{-1}\alpha'_{-1} + \lambda\alpha''_1,$$

where α_j is the j -th eigenspace of the derivative of σ , ($j = -1, 0, 1$). The following is a well known fact, see for example, [3].

Theorem 1.6. *Let M be a surface and N a semisimple k -symmetric space ($k > 2$). If a map $\varphi : M \rightarrow N$ is a (Lorentz) primitive map then $d + \alpha^{\lambda}$ is a family of flat connections.*

2. PROJECTIVE MINIMAL SURFACES AND DEMOULIN SURFACES

2.1. Projective minimal surfaces. Let τ_2 be the outer involution on $\mathrm{SL}_4\mathbb{R}$ associated to the symmetric space Q_2 in (1.11) defined by $\tau_2(X) = J_2 X^{t-1} J_2$, $X \in \mathrm{SL}_4\mathbb{R}$ and $J_2 = \text{offdiag}(1, -1, -1, 1)$. Abuse of notation, we denote the differential of τ_2 by the same symbol τ_2 which is an outer involution on $\mathrm{sl}_4\mathbb{R}$:

$$(2.1) \quad \tau_2(X) = -J_2 X^t J_2, \quad X \in \mathrm{sl}_4\mathbb{R}.$$

Let us consider the eigenspace decomposition of $\mathfrak{g} = \mathrm{sl}_4\mathbb{R}$ with respect to τ_2 , that is, $\mathfrak{g} = \mathfrak{k}_2 \oplus \mathfrak{p}_2$, where \mathfrak{k}_2 is the 0th-eigenspace and \mathfrak{p}_2 is the 1st-eigenspace as follows:

$$\mathfrak{k}_2 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & 0 & a_{13} \\ a_{31} & 0 & -a_{22} & a_{12} \\ 0 & a_{31} & a_{21} & -a_{11} \end{pmatrix} \middle| a_{ij} \in \mathbb{R} \right\}, \quad \mathfrak{p}_2 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & -a_{11} & a_{23} & -a_{13} \\ a_{31} & a_{32} & -a_{11} & -a_{12} \\ a_{41} & -a_{31} & -a_{21} & a_{11} \end{pmatrix} \middle| a_{ij} \in \mathbb{R} \right\}.$$

According to this decomposition $\mathfrak{g} = \mathfrak{k}_2 \oplus \mathfrak{p}_2$, the Maurer-Cartan form $\alpha = F^{-1}dF = Udx + Vdy$ can be decomposed into

$$\alpha = \alpha_{\mathfrak{k}_2} + \alpha_{\mathfrak{p}_2} = U_{\mathfrak{k}_2}dx + V_{\mathfrak{k}_2}dy + U_{\mathfrak{p}_2}dx + V_{\mathfrak{p}_2}dy,$$

where $U = U_{\mathfrak{k}_2} + U_{\mathfrak{p}_2}$ and $V = V_{\mathfrak{k}_2} + V_{\mathfrak{p}_2}$. Let us insert the parameter $\lambda \in \mathbb{R}^\times$ into U and V as follows:

$$U^\lambda = U_{\mathfrak{k}_2} + \lambda^{-1}U_{\mathfrak{p}_2} \text{ and } V^\lambda = V_{\mathfrak{k}_2} + \lambda V_{\mathfrak{p}_2}.$$

Then a family of 1-forms α^λ is defined as follows:

$$(2.2) \quad \alpha^\lambda = \alpha_{\mathfrak{k}_2} + \lambda^{-1}\alpha'_{\mathfrak{p}_2} + \lambda\alpha''_{\mathfrak{p}_2} = U^\lambda dx + V^\lambda dy.$$

In fact the matrices U^λ and V^λ are explicitly given as follows:

$$(2.3) \quad U^\lambda = \begin{pmatrix} \frac{c_x}{2c} & P & k & \lambda^{-1}bQ \\ 1 & -\frac{c_x}{2c} & 0 & k \\ 0 & \lambda^{-1}b & \frac{c_x}{2c} & P \\ 0 & 0 & 1 & -\frac{c_x}{2c} \end{pmatrix}, \quad V^\lambda = \begin{pmatrix} \frac{b_y}{2b} & \ell & Q & \lambda cP \\ 0 & \frac{b_y}{2b} & \lambda c & Q \\ 1 & 0 & -\frac{b_y}{2b} & \ell \\ 0 & 1 & 0 & -\frac{b_y}{2b} \end{pmatrix}.$$

The following is one of the main theorems in this paper.

Theorem 2.1. *Let S be a surface in \mathbb{P}^3 and g_2 the second-order Gauss map defined in (1.11). Moreover, let α^λ ($\lambda \in \mathbb{R}^\times$) be a family of 1-forms defined in (2.2). Then the followings are mutually equivalent:*

- (1) *The surface S is a projective minimal surface.*
- (2) *The second-order Gauss map g_2 is a conformal Lorentz harmonic map into Q_2 .*
- (3) *$d + \alpha^\lambda$ is a family of flat connections on $\mathbb{D} \times \text{SL}_4\mathbb{R}$.*

Proof. Let us compute the flatness conditions of $d + \alpha^\lambda$, that is, the Maurer-Cartan equation $d\alpha^\lambda + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0$. It is easy to see that except the (1, 4)-entry, the Maurer-Cartan equation is equivalent to (1.6). Moreover, the λ^{-1} -term and the λ -term of the (1, 4)-entry are equivalent to that the first equation and the second equation in (1.8), respectively. Thus the equivalence of (1) and (3) follows.

The equivalence of (2) and (3) follows from Theorem 1.4, since the family of 1-forms α^λ is given by the involution τ_2 and it defines the symmetric space $Q_2 = \text{SL}_4\mathbb{R}/K_2$. \square

The above theorem implies that if S is a projective minimal surface, then there exists a family of projective minimal surface S^λ ($\lambda \in \mathbb{R}^\times$) such that $S^\lambda|_{\lambda=1} = S$. Projective minimal surfaces of the family have the same projective metric $8bc dx dy$ but the different conformal classes of cubic forms $\lambda^{-1}b dx^3 + \lambda c dy^3$. Thus the family of the Maurer-Cartan form α^λ defines a family of Wilczynski frames F^λ such that $(F^\lambda)^{-1}dF^\lambda = \alpha^\lambda$. It is easy to see that F^λ is an element of the twisted loop group of $\text{SL}_4\mathbb{R}$:

$$\text{ASL}_4\mathbb{R}_{\tau_2} = \{g : \mathbb{R}^\times \rightarrow \text{SL}_4\mathbb{R} \mid \tau_2 g(\lambda) = g(-\lambda)\}.$$

This family of Wilczynski frames F^λ will be called the *extended Wilczynski frame* for a projective minimal surface.

Remark 2.2. Using the Plücker embedding from \mathbb{P}^3 to \mathbb{P}^5 , it was shown that a surface is projective minimal if and only if the conformal Gauss map is conformal Lorentz harmonic, see [2, Theorem 7]. This is a similar to the equivalence (1) and (2) in Theorem 2.1, however, the conformal Gauss map takes values in the Grassmannian.

2.2. Demoulin surfaces. Let τ_1 be the outer involution on $\mathrm{SL}_4\mathbb{R}$ associated to Q_1 in (1.10) defined by $\tau_1(X) = J_1 X^{t-1} J_1$, $X \in \mathrm{SL}_4\mathbb{R}$ and $J_1 = \text{offdiag}(1, -1, -1, 1)$. Abuse of notation, we denote the differential of τ_1 by the same symbol τ_1 which is an outer involution on $\mathrm{sl}_4\mathbb{R}$:

$$(2.4) \quad \tau_1(X) = -J_1 X^t J_1, \quad X \in \mathrm{sl}_4\mathbb{R}.$$

Let us consider the eigenspace decomposition of $\mathfrak{g} = \mathrm{sl}_4\mathbb{R}$ with respect to τ_1 , that is, $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1$, where \mathfrak{k}_1 is the 0th-eigenspace and \mathfrak{p}_1 is the 1st-eigenspace as follows:

$$\mathfrak{k}_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & 0 & -a_{13} \\ a_{31} & 0 & -a_{22} & -a_{12} \\ 0 & -a_{31} & -a_{21} & -a_{11} \end{pmatrix} \middle| a_{ij} \in \mathbb{R} \right\}, \quad \mathfrak{p}_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & -a_{11} & a_{23} & a_{13} \\ a_{31} & a_{32} & -a_{11} & a_{12} \\ a_{41} & a_{31} & a_{21} & a_{11} \end{pmatrix} \middle| a_{ij} \in \mathbb{R} \right\}.$$

According to this decomposition $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1$, the Maurer-Cartan form $\alpha = F^{-1}dF = Udx + Vdy$ can be decomposed into

$$\alpha = \alpha_{\mathfrak{k}_1} + \alpha_{\mathfrak{p}_1} = U_{\mathfrak{k}_1}dx + V_{\mathfrak{k}_1}dy + U_{\mathfrak{p}_1}dx + V_{\mathfrak{p}_1}dy,$$

where $U = U_{\mathfrak{k}_1} + U_{\mathfrak{p}_1}$ and $V = V_{\mathfrak{k}_1} + V_{\mathfrak{p}_1}$. Let us insert the parameter $\lambda \in \mathbb{R}^\times$ into U and V as follows:

$$\tilde{U}^\lambda = U_{\mathfrak{k}_1} + \lambda^{-1}U_{\mathfrak{p}_1} \quad \text{and} \quad \tilde{V}^\lambda = V_{\mathfrak{k}_1} + \lambda V_{\mathfrak{p}_1}.$$

Then a family of 1-forms $\tilde{\alpha}_\lambda$ is defined as follows:

$$(2.5) \quad \tilde{\alpha}^\lambda = \alpha_{\mathfrak{k}_1} + \lambda^{-1}\alpha'_{\mathfrak{p}_1} + \lambda\alpha''_{\mathfrak{p}_1} = \tilde{U}^\lambda dx + \tilde{V}^\lambda dy.$$

In fact the matrices \tilde{U}^λ and \tilde{V}^λ are explicitly given as follows:

$$(2.6) \quad \tilde{U}^\lambda = \begin{pmatrix} \frac{c_x}{2c} & \lambda^{-1}P & \lambda^{-1}k & \lambda^{-1}bQ \\ \lambda^{-1} & -\frac{c_x}{2c} & 0 & \lambda^{-1}k \\ 0 & \lambda^{-1}b & \frac{c_x}{2c} & \lambda^{-1}P \\ 0 & 0 & \lambda^{-1} & -\frac{c_x}{2c} \end{pmatrix}, \quad \tilde{V}^\lambda = \begin{pmatrix} \frac{b_y}{2b} & \lambda\ell & \lambda Q & \lambda cP \\ 0 & \frac{b_y}{2b} & \lambda c & \lambda Q \\ \lambda & 0 & -\frac{b_y}{2b} & \lambda\ell \\ 0 & \lambda & 0 & -\frac{b_y}{2b} \end{pmatrix}.$$

The following is one of the main theorems in this paper.

Theorem 2.3. *Let S be a surface in \mathbb{P}^3 and g_1 the first-order Gauss map defined in (1.10). Moreover, let $\tilde{\alpha}^\lambda$ ($\lambda \in \mathbb{R}^\times$) be a family of 1-forms defined in (2.5). Then the followings are mutually equivalent:*

- (1) *The surface S is a Demoulin surface or a projective minimal coincidence surface.*
- (2) *The first-order Gauss map g_1 is a Lorentz harmonic map into Q_1 .*
- (3) *$d + \tilde{\alpha}^\lambda$ is a family of flat connections on $\mathbb{D} \times \mathrm{SL}_4\mathbb{R}$.*

Proof. Let us compute the flatness conditions of $d + \tilde{\alpha}^\lambda$, that is, the Maurer-Cartan equation $d\tilde{\alpha}^\lambda + \frac{1}{2}[\tilde{\alpha}^\lambda \wedge \tilde{\alpha}^\lambda] = 0$. A straightforward computation shows that these are equivalent to

$$\begin{aligned} Q_x = P_y &= 0, \quad k_y + k \frac{b_y}{b} = 0, \quad \ell_x + \ell \frac{c_x}{c} = 0, \\ bQ_y + 2b_y Q &= 0, \quad cP_x + 2c_x P = 0. \end{aligned}$$

The surfaces with $P = Q = 0$ satisfies the above equations and they are Demoulin surfaces by (1.9). Assume that $P \neq 0$ (The case of $Q \neq 0$ is similar). From the first equation and the last equation, P and $(\log c)_x$ depend only on x . Moreover from the equation $\ell_x + \ell(\log c)_x = 0$ and the definition ℓ in (1.4), $(\log b)_x = -2(\log c)_x$ and thus $(\log b)_x$ depends only on x . Thus $(\log b/c)_{xy} = 0$, which means that it is an *isothermally asymptotic surface*. Using a scaling transformation and a change of coordinates, we can assume that $b = c$. Then $\ell = k$ and the equations $\ell_x + \ell(\log c)_x = 0$ and $k_y + k(\log b)_y = 0$ imply that $b (= c)$ is constant. Thus P and Q are constant, and from (1.5) $p \neq 0$ and q are constant. Therefore, the canonical system is given by

$$f_{xx} = f_y + pf, \quad f_{yy} = f_y + qf.$$

A surface satisfying the above equation is the special case of the *coincidence surface*, [11, Example 2.19]. In fact, it is easy to see that the surface is a projective minimal coincidence surface. Thus the equivalence of (1) and (3) follows.

The equivalence of (2) and (3) follows from Theorem 1.4, since the family of 1-forms $\tilde{\alpha}^\lambda$ is given by the involution τ_1 and it defines the symmetric space $Q_1 = \mathrm{SL}_4\mathbb{R}/K_1$. \square

Remark 2.4.

- (1) As we have seen, Demoulin surfaces and projective minimal coincidence surfaces give a special class of projective minimal surfaces. For the Gauss maps, this implies that the Lorentz harmonicity of the first-order Gauss map induces the Lorentz harmonicity of the second-order Gauss map.
- (2) Let \tilde{F}^λ be a family of frames such that $(\tilde{F}^\lambda)^{-1}d\tilde{F}^\lambda = \tilde{\alpha}^\lambda$. It is easy to see from the forms of \tilde{U}^λ and \tilde{V}^λ in (2.6) that \tilde{F}^λ is not the Wilczynski frame of a Demoulin surface or projective minimal coincidence surface except $\lambda = 1$. However conjugating \tilde{F}^λ by $D\tilde{F}^\lambda D^{-1}$ with $D = \mathrm{diag}(1, \lambda, \lambda^{-1}, 1)$, the frames $D\tilde{F}^\lambda D^{-1}$ give a family of Wilczynski frames for Demoulin surfaces or projective minimal coincidence surfaces. The corresponding Demoulin surfaces or projective minimal coincidence surfaces have the same projective metric $8bcdxdy$ but the different conformal classes of cubic forms $\lambda^{-3}b dx^3 + \lambda^3c dy^3$. Moreover, the functions P and Q change as $\lambda^{-2}P$ and λ^2Q , respectively.

Corollary 2.5. *Retaining the assumptions in Theorem 2.3, the followings are equivalent:*

- (1) *The surface S is a Demoulin surface.*
- (2) *The first-order Gauss map g_1 is a conformal Lorentz harmonic map into Q_1 .*

Proof. From Proposition 1.2, it is easy to see that the first-order Gauss map is conformal if and only if it satisfies that $P = Q = 0$, that is, the surface is a Demoulin surface. Moreover, from Theorem 2.3 the Gauss map of the Demoulin surface is Lorentz harmonic. \square

Let S be a Demoulin surface or projective minimal coincidence surface and \tilde{F}^λ a family of frames such that $(\tilde{F}^\lambda)^{-1}d\tilde{F}^\lambda = \tilde{\alpha}^\lambda$. Then \tilde{F}^λ will be called the *extended Wilczynski frame* for a Demoulin surface or projective minimal coincidence surface.

We now show that the extended Wilczynski frame for a Demoulin surface has an additional order three cyclic symmetry. Let σ be an order three automorphism on the complexification of $\mathrm{SL}_4\mathbb{R}$ as follows:

$$\sigma g = \mathrm{Ad}(E)g, \quad g \in \mathrm{SL}_4\mathbb{C},$$

where $E = \mathrm{diag}(1, \epsilon^2, \epsilon, 1)$ with $\epsilon = e^{2\pi i/3}$. Then it is easy to see that $\tilde{F}(\lambda) (= \tilde{F}^\lambda)$ satisfies $\sigma\tilde{F}(\lambda) = \tilde{F}(\epsilon\lambda)$, since $\tilde{U}(\lambda) (= \tilde{U}^\lambda)$ and $\tilde{V}(\lambda) (= \tilde{V}^\lambda)$ satisfy the same symmetry. It is also easy to see that τ_1 and σ commute, and $\kappa = \tau_1 \circ \sigma$ defines an order six automorphism. Thus, the extended Wilczynski frame $\tilde{F}(\lambda)$ satisfies

$$\kappa\tilde{F}(\lambda) = \tilde{F}(-\epsilon\lambda).$$

Note that $-\epsilon$ is the 6th root of unity. From the above argument, it is easy to see that the extended Wilczynski frame $\tilde{F}(\lambda)$ for a Demoulin surface is an element of the twisted loop group of $\mathrm{SL}_4\mathbb{R}$:

$$\Lambda\mathrm{SL}_4\mathbb{R}_\kappa = \{g : \mathbb{R}^\times \rightarrow \mathrm{SL}_4\mathbb{R} \mid \kappa g(\lambda) = g(-\epsilon\lambda)\}.$$

Corollary 2.6. *The first-order Gauss map of a Demoulin surface, which is conformal and Lorentz harmonic in $Q_1 = \mathrm{SL}_4\mathbb{R}/K_1$, can be obtained by the natural projection of the Lorentz primitive map into the 6-symmetric space $\mathrm{SL}_4\mathbb{C}/K$ with $K = \{\mathrm{diag}(k_1, k_2, k_2^{-1}, k_1^{-1}) \mid k_1, k_2 \in \mathbb{C}^\times\}$.*

Proof. The -1 st-eigenspace and the 1 st-eigenspace of the derivative of the order six automorphism $\kappa = \tau_2 \circ \sigma$ are described as follows:

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & a_{13} & 0 \\ a_{21} & 0 & 0 & a_{24} \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & a_{43} & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ 0 & 0 & a_{23} & 0 \\ a_{31} & 0 & 0 & a_{34} \\ 0 & a_{42} & 0 & 0 \end{pmatrix} \right\}.$$

From the matrices \tilde{U}^λ and \tilde{V}^λ in (2.6) with $P = Q = 0$, we see that the conditions in (1.12) of a Lorentz primitive map are satisfied. The stabilizer of κ is

$$K = \{\mathrm{diag}(k_1, k_2, k_2^{-1}, k_1^{-1}) \mid k_1, k_2 \in \mathbb{C}^\times\}.$$

Therefore there is a Lorentz primitive map $g = FJF^t$ with $J = KJ_1$ into the 6-symmetric space $\mathrm{SL}_4\mathbb{C}/K$ such that $\pi \circ g = g_1$, where π is the natural projection $\pi : \mathrm{SL}_4\mathbb{C}/K \rightarrow \mathrm{SL}_4\mathbb{C}/K_1^\mathbb{C}$. \square

Remark 2.7. Since we obtained the Lorentz harmonic Gauss map into the symmetric space for a projective minimal surface and the Lorentz primitive map into the 6-symmetric space for a Demoulin surface, the generalized Weierstrass type representation as in [5, 6] can be established.

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